

ON GENERALIZED SPHERICAL SURFACES IN EUCLIDEAN SPACES

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Abstract

In the present study we consider the generalized rotational surfaces in Euclidean spaces. Firstly, we consider generalized spherical curves in Euclidean $(n+1)$ -space \mathbb{E}^{n+1} . Further, we introduce some kind of generalized spherical surfaces in Euclidean spaces \mathbb{E}^3 and \mathbb{E}^4 respectively. We have shown that the generalized spherical surfaces of first kind in \mathbb{E}^4 are known as rotational surfaces, and the second kind generalized spherical surfaces are known as meridian surfaces in \mathbb{E}^4 . We have also calculated the Gaussian, normal and mean curvatures of these kind of surfaces. Finally, we give some examples.

1 Introduction

¹The Gaussian curvature and mean curvature of the surfaces in Euclidean spaces play an important role in differential geometry. Especially, surfaces with constant Gaussian curvature [18], and constant mean curvature conform nice classes of surfaces which are important for surface modelling [3]. Surfaces with constant negative curvature are known as pseudo-spherical surfaces [14].

Rotational surfaces in Euclidean spaces are also important subject of differential geometry. The rotational surfaces in \mathbb{E}^3 are called surface of revolution. Recently V. Velickovic classified all rotational surfaces in \mathbb{E}^3 with constant Gaussian curvature [17]. Rotational surfaces in \mathbb{E}^4 was first introduced by C. Moore in 1919. In the recent years some mathematicians have taken an interest in the rotational surfaces in \mathbb{E}^4 , for example G. Ganchev and V. Milousheva [12], U. Dursun and N. C. Turgay [11], K. Arslan, et al. [1] and D.W.Yoon [19]. In [12], the authors applied invariance theory of surfaces in the four dimensional Euclidean space to the class of general rotational surfaces whose meridians lie in two-dimensional planes in order to find all minimal super-conformal surfaces. These surfaces were further studied in [11], which found all minimal surfaces by solving the differential equation that characterizes minimal surfaces. They then

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determined all pseudo-umbilical general rotational surfaces in \mathbb{E}^4 . K. Arslan et.al in [1] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical, they also shown that each general rotational surface is a Chen surface in \mathbb{E}^4 and gave some special classes of generalized rotational surfaces as examples. See also [9] and [2] rotational surfaces with Constant Gaussian Curvature in Four-Space. For higher dimensional case N.H. Kuiper defined rotational embedded submanifolds in Euclidean spaces [15].

The meridian surfaces in \mathbb{E}^4 was first introduced by G. Ganchev and V. Milousheva (See, [13] and [5]) which are the special kind of rotational surfaces. Basic source of examples of surfaces in 4-dimensional Euclidean or pseudo-Euclidean space are the standard rotational surfaces and the general rotational surfaces. Further, Ganchev and Milousheva defined another class of surfaces of rotational type which are one-parameter system of meridians of a rotational hypersurface. They constructed a family of surfaces with at normal connection lying on a standard rotational hypersurface in \mathbb{E}^4 as a meridian surfaces. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two dimensional axis in \mathbb{E}^4 .

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in \mathbb{E}^n . Section 3 explains some geometric properties of spherical curves \mathbb{E}^{n+1} . Section 4 tells about the generalized spherical surfaces in \mathbb{E}^{n+m} . Further this section provides some basic properties of generalized spherical surfaces in \mathbb{E}^4 and the structure of their curvatures. We also shown that every generalized spherical surfaces in \mathbb{E}^4 have constant Gaussian curvature $K = 1/c^2$. Finally, we present some examples of generalized spherical surfaces in \mathbb{E}^4 .

2 Basic Concepts

Let M be a smooth surface in \mathbb{E}^n given with the patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u, v)$ of M span $\{X_u, X_v\}$. In the chart (u, v) the coefficients of the first fundamental form of M are given by

$$g_{11} = \langle X_u, X_u \rangle, g_{12} = \langle X_u, X_v \rangle, g_{22} = \langle X_v, X_v \rangle, \quad (1)$$

where \langle, \rangle is the Euclidean inner product. We assume that $W^2 = g_{11}g_{22} - g_{12}^2 \neq 0$, i.e. the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^\perp M$ where $T_p^\perp M$ is the orthogonal component of T_pM in \mathbb{E}^n .

Let $\chi(M)$ and $\chi^\perp(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M , respectively. Given any local vector fields X_1, X_2 tangent to M , consider the second fundamental map $h : \chi(M) \times \chi(M) \rightarrow \chi^\perp(M)$;

$$h(X_i, X_j) = \tilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2 \quad (2)$$

where ∇ and $\tilde{\nabla}$ are the induced connection of M and the Riemannian connection of \mathbb{E}^n , respectively. This map is well-defined, symmetric and bilinear [6].

For any arbitrary orthonormal frame field $\{N_1, N_2, \dots, N_{n-2}\}$ of M , recall the shape operator $A : \chi^\perp(M) \times \chi(M) \rightarrow \chi(M)$;

$$A_{N_k} X_j = -(\tilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M). \quad (3)$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = L_{ij}^k, \quad 1 \leq i, j \leq 2; \quad 1 \leq k \leq n-2 \quad (4)$$

where L_{ij}^k are the coefficients of the second fundamental form. The equation (2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} L_{ij}^k N_k, \quad 1 \leq i, j \leq 2 \quad (5)$$

holds. Then the Gauss curvature K of a regular patch $X(u, v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{n-2} (L_{11}^k L_{22}^k - (L_{12}^k)^2). \quad (6)$$

Further, the mean curvature vector of a regular patch $X(u, v)$ is given by

$$\vec{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12}) N_k. \quad (7)$$

We call the functions

$$H_k = \frac{(L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12})}{2W^2}, \quad (8)$$

the k .th mean curvature functions of the given surface. The norm of the mean curvature vector $H = \|\vec{H}\|$ is called the mean curvature of M . Recall that a surface M is said to be *flat* (resp. *minimal*) if its Gauss curvature (resp. mean curvature vector) vanishes identically [7], [8].

The normal curvature K_N of M is defined by (see [10])

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \langle R^\perp(X_1, X_2) N_\alpha, N_\beta \rangle^2 \right\}^{1/2}. \quad (9)$$

where

$$R^\perp(X_i, X_j) N_\alpha = h(X_i, A_{N_\alpha} X_j) - h(X_j, A_{N_\alpha} X_i), \quad (10)$$

and

$$\langle R^\perp(X_i, X_j) N_\alpha, N_\beta \rangle = \langle [A_{N_\alpha}, A_{N_\beta}] X_i, X_j \rangle, \quad (11)$$

is called the *equation of Ricci*. We observe that the normal connection D of M is flat if and only if $K_N = 0$, and by a result of Cartan, this equivalent to the diagonalisability of all shape operators A_{N_α} [6].

3 Generalized Spherical Curves

Let γ be a regular oriented curve in \mathbb{E}^{n+1} that does not lie in any subspace of \mathbb{E}^{n+1} . From each point of the curve γ one can draw a segment of unit length along the normal line corresponding to the chosen orientation. The ends of these segments describe a new curve β . The curve $\gamma \in \mathbb{E}^{n+1}$ is called a *generalized spherical curve* if the curve β lies in a certain subspace \mathbb{E}^n of \mathbb{E}^{n+1} . The curve β is called the trace of γ [14]. Let

$$\gamma(u) = (f_1(u), \dots, f_{n+1}(u)), \quad (12)$$

be the radius vector of the curve γ given with arclength parametrization u , i.e., $\|\gamma'(u)\| = 1$. The curve β is defined by the radius vector

$$\beta(u) = (\gamma + c^2 \gamma'')(u) = ((f_1 + c^2 f_1'')(u), \dots, (f_{n+1} + c^2 f_{n+1}'')(u)), \quad (13)$$

where c is a real constant. If γ is a generalized spherical curve of \mathbb{E}^{n+1} then by definition the curve β lies in the hyperplane \mathbb{E}^n if and only if $f_{n+1} + c^2 f_{n+1}'' = 0$. Consequently, this equation has a non-trivial solution

$$f_{n+1}(u) = \lambda \cos\left(\frac{u}{c} + c_0\right),$$

with some constants λ and c_0 . By a suitable choice of arclength we may assume that

$$f_{n+1}(u) = \lambda \cos\left(\frac{u}{c}\right), \quad (14)$$

with $\lambda > 0$. Thus, the radius vector of the generalized spherical curve γ takes the form

$$\gamma(u) = \left(f_1(u), \dots, f_n(u), \lambda \cos\left(\frac{u}{c}\right)\right). \quad (15)$$

Moreover, the condition for the arclength parameter u implies that

$$(f_1')^2 + \dots + (f_n')^2 = 1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right). \quad (16)$$

For convenience, we introduce a vector function

$$\phi(u) = (f_1(u), \dots, f_n(u); 0).$$

Then the radius vector (15) can be represented in the form

$$\gamma(u) = \phi(u) + \lambda \cos\left(\frac{u}{c}\right) e_{n+1}, \quad (17)$$

where $e_{n+1} = (0, 0, \dots, 0, 1)$. Consequently, the condition (16) gives

$$\|\phi'(u)\|^2 = 1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right). \quad (18)$$

Hence, the radius vector of the trace curve β becomes

$$\beta(u) = \phi(u) + c^2 \phi''(u). \quad (19)$$

Consider an arbitrary unit vector function

$$a(u) = (a_1(u), \dots, a_n(u); 0), \quad (20)$$

in \mathbb{E}^{n+1} and use this function to construct a new vector function

$$\phi(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} a(u) du, \quad (21)$$

whose last coordinate is equal to zero. Consequently, the vector function $\phi(u)$ satisfies the condition (18) and generates a generalized spherical curve with radius vector (17).

Example 1 *The ordinary circular curve in \mathbb{E}^2 is given with the radius vector*

$$\gamma(u) = \left(\int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} du, \lambda \cos \left(\frac{u}{c} \right) \right). \quad (22)$$

Example 2 *Consider the unit vector $a(u) = (\cos \alpha(u), \sin \alpha(u); 0)$ in \mathbb{E}^2 . Then using (21), the corresponding generalized spherical curve in \mathbb{E}^3 is defined by the radius vector*

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} \cos \alpha(u) du, \\ f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} \sin \alpha(u) du, \\ f_3(u) &= \lambda \cos \left(\frac{u}{c} \right). \end{aligned} \quad (23)$$

Example 3 *Consider the unit vector*

$$a(u) = (\cos \alpha(u), \cos \alpha(u) \sin \alpha(u), \sin^2 \alpha(u); 0)$$

in \mathbb{E}^3 . Then using (21), the corresponding generalized spherical curve in \mathbb{E}^4 is defined by the radius vector

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} \cos \alpha(u) du, \\ f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} \cos \alpha(u) \sin \alpha(u) du, \\ f_3(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \left(\frac{u}{c} \right)} \sin^2 \alpha(u) du; \\ f_4(u) &= \lambda \cos \left(\frac{u}{c} \right). \end{aligned} \quad (24)$$

4 Generalized Spherical Surfaces

Consider the space $\mathbb{E}^{n+1} = \mathbb{E}^n \oplus \mathbb{E}^1$ as a subspace of $\mathbb{E}^{n+m} = \mathbb{E}^n \oplus \mathbb{E}^m$, $m \geq 2$ and Cartesian coordinates x_1, x_2, \dots, x_{n+m} and orthonormal basis e_1, \dots, e_{n+m} in \mathbb{E}^{n+m} . Let M^2 be a local surface given with the regular patch (radius vector) $\mathbb{E}^n \subset \mathbb{E}^{n+1}$

$$X(u, v) = \phi(u) + \lambda \cos\left(\frac{u}{c}\right) \rho(v), \quad (25)$$

where the vector function $\phi(u) = (f_1(u), \dots, f_n(u), 0, \dots, 0)$, satisfies (18) and generates a generalized spherical curve with radius vector

$$\gamma(u) = \phi(u) + \lambda \cos\left(\frac{u}{c}\right) e_{n+1}, \quad (26)$$

and the vector function $\rho(v) = (0, \dots, 0, g_1(v), \dots, g_m(v))$, satisfying the conditions $\|\rho(v)\| = 1$, $\|\rho'(v)\| = 1$, and specifies a curve $\rho = \rho(v)$ parametrized by a natural parameter on the unit sphere $S^{m-1} \subset \mathbb{E}^m$. Consequently, the surface M^2 is obtained as a result of the rotation of the generalized spherical curve γ along the spherical curve ρ , which is called *generalized Spherical surface* in \mathbb{E}^{n+m} .

In the sequel, we will consider some type of generalized spherical surface;

CASE I. For $n = 1$ and $m = 2$, the radius vector (25) satisfying the indicated properties describes the *spherical surface* in \mathbb{E}^3 with the radius vector

$$X(u, v) = (\phi(u), \lambda \cos\left(\frac{u}{c}\right) \cos v, \lambda \cos\left(\frac{u}{c}\right) \sin v), \quad (27)$$

where the function $\phi(u)$ is found from the relation $|\phi'(u)| = \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)}$. The surface given with the parametrization (27) is a kind of surface of revolution which is called ordinary sphere.

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u(u, v) &= \left(\phi'(u), \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v\right), \\ X_v(u, v) &= \left(0, -\lambda \cos\left(\frac{u}{c}\right) \sin v, \lambda \cos\left(\frac{u}{c}\right) \cos v\right). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u(u, v), X_u(u, v) \rangle = 1 \\ g_{12} &= \langle X_u(u, v), X_v(u, v) \rangle = 0 \\ g_{22} &= \langle X_v(u, v), X_v(u, v) \rangle = \lambda^2 \cos^2\left(\frac{u}{c}\right), \end{aligned}$$

where \langle, \rangle is the standard scalar product in \mathbb{E}^3 .

For a regular patch $X(u, v)$ the unit normal vector field or surface normal N is defined by

$$\begin{aligned} N(u, v) &= \frac{X_u \times X_v}{\|X_u \times X_v\|}(u, v) \\ &= \left(-\frac{\lambda}{c} \sin\left(\frac{u}{c}\right), -\phi'(u) \cos v, -\phi'(u) \sin v \right), \end{aligned}$$

where

$$\|X_u \times X_v\| = \sqrt{g_{11}g_{22} - g_{12}^2} = \lambda \cos\left(\frac{u}{c}\right) \neq 0.$$

The second partial derivatives of $X(u, v)$ are expressed as follows

$$\begin{aligned} X_{uu}(u, v) &= \left(\phi''(u), \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \sin v \right), \\ X_{uv}(u, v) &= \left(0, \frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v, -\frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \cos(v) \right), \\ X_{vv}(u, v) &= \left(0, -\lambda \cos\left(\frac{u}{c}\right) \cos v, -\lambda \cos\left(\frac{u}{c}\right) \sin(v) \right). \end{aligned}$$

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{aligned} L_{11} &= \langle X_{uu}(u, v), N(u, v) \rangle = -\kappa_1(u), \\ L_{12} &= \langle X_{uv}(u, v), N(u, v) \rangle = 0, \\ L_{22} &= \langle X_{vv}(u, v), N(u, v) \rangle = \phi'(u) \lambda \cos\left(\frac{u}{c}\right) \end{aligned} \quad (28)$$

where

$$\kappa_1(u) = -\frac{\lambda}{c^2} \phi'(u) \cos\left(\frac{u}{c}\right) + \frac{\lambda}{c} \phi''(u) \sin\left(\frac{u}{c}\right), \quad (29)$$

is the differentiable function. Furthermore, substituting (28) into (6)-(7) we obtain the following result.

Proposition 4 *Let M be a spherical surface in \mathbb{E}^3 given with the parametrization (27). Then the Gaussian and mean curvature of M become*

$$K = 1/c^2,$$

and

$$H = \frac{\frac{2\lambda^2}{c^2} \cos^2\left(\frac{u}{c}\right) - \frac{\lambda^2}{c^2} + 1}{2\lambda \cos\left(\frac{u}{c}\right) \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)}},$$

respectively.

Corollary 5 [17] *Let M be a spherical surface in \mathbb{E}^3 given with the parametrization (27). Then we have the following asertations*

- i) *If $\lambda = c$ then the corresponding surface is a sphere with radius c and centered at the origin,*
- ii) *If $\lambda > c$ then the corresponding surface is a hyperbolic spherical surface,*
- iii) *If $\lambda < c$ then the corresponding surface is an elliptic spherical surface.*

CASE II. For $n = 2$ and $m = 2$, the radius vector (25) satisfying the indicated properties describes the *generalized spherical surface* given with the radius vector

$$X(u, v) = (f_1(u), f_2(u), \lambda \cos\left(\frac{u}{c}\right) \cos v, \lambda \cos\left(\frac{u}{c}\right) \sin v), \quad (30)$$

where

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} \cos \alpha(u) du, \\ f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} \sin \alpha(u) du. \end{aligned} \quad (31)$$

are differentiable functions.

We call this surface the *generalized spherical surface of first kind*. Actually, these surfaces are the special type of rotational surfaces [12], see also [2].

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u(u, v) &= (f_1'(u), f_2'(u), \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v), \\ X_v(u, v) &= (0, 0, -\lambda \cos\left(\frac{u}{c}\right) \sin v, \lambda \cos\left(\frac{u}{c}\right) \cos v). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u(u, v), X_u(u, v) \rangle = 1 \\ g_{12} &= \langle X_u(u, v), X_v(u, v) \rangle = 0 \\ g_{22} &= \langle X_v(u, v), X_v(u, v) \rangle = \lambda^2 \cos^2\left(\frac{u}{c}\right), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{E}^4 .

The second partial derivatives of $X(u, v)$ are expressed as follows

$$\begin{aligned} X_{uu}(u, v) &= (f_1''(u), f_2''(u), \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \sin v), \\ X_{uv}(u, v) &= (0, 0, \frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v, -\frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \cos v), \\ X_{vv}(u, v) &= (0, 0, -\lambda \cos\left(\frac{u}{c}\right) \cos v, -\lambda \cos\left(\frac{u}{c}\right) \sin v). \end{aligned}$$

The normal space is spanned by the vector fields

$$\begin{aligned} N_1 &= \frac{1}{\kappa} (f_1''(u), f_2''(u), \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \sin v) \\ N_2 &= \frac{1}{\kappa} \left(\frac{-\lambda f_2'(u)}{c^2} \cos\left(\frac{u}{c}\right) + \frac{\lambda f_2''(u)}{c} \sin\left(\frac{u}{c}\right), \frac{-\lambda f_1''(u)}{c} \sin\left(\frac{u}{c}\right) + \frac{\lambda f_1'(u)}{c^2} \cos\left(\frac{u}{c}\right), \right. \\ &\quad \left. (f_1'(u) f_2''(u) - f_1''(u) f_2'(u)) \cos v, (f_1'(u) f_2''(u) - f_1''(u) f_2'(u)) \sin v \right) \end{aligned}$$

where

$$\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + \frac{\lambda^2}{c^4} \cos^2\left(\frac{u}{c}\right)}, \quad (32)$$

is the curvature of the profile curve γ . Hence, the coefficients of the second fundamental form of the surface are

$$\begin{aligned}
L_{11}^1 &= \langle X_{uu}(u, v), N_1(u, v) \rangle = \kappa(u), \\
L_{12}^1 &= \langle X_{uv}(u, v), N_1(u, v) \rangle = 0, \\
L_{22}^1 &= \langle X_{vv}(u, v), N_1(u, v) \rangle = \frac{\lambda^2 \cos^2\left(\frac{u}{c}\right)}{c^2 \kappa(u)}, \\
L_{11}^2 &= \langle X_u(u, v), N_2(u, v) \rangle = 0, \\
L_{12}^2 &= \langle X_{uv}(u, v), N_2(u, v) \rangle = 0, \\
L_{22}^2 &= \langle X_{vv}(u, v), N_2(u, v) \rangle = -\frac{\lambda \cos\left(\frac{u}{c}\right) \kappa_1(u)}{\kappa(u)}.
\end{aligned} \tag{33}$$

where

$$\kappa_1(u) = f_1'(u)f_2''(u) - f_1''(u)f_2'(u), \tag{34}$$

is the differentiable function.

Furthermore, by the use of (33) with (6)-(7) we obtain the following results.

Proposition 6 *The generalized spherical surface of first kind has constant Gaussian curvature $K = 1/c^2$.*

Proposition 7 *Let M be a generalized spherical surface of first kind given with the surface patch (30). Then the mean curvature vector of M becomes*

$$\vec{H} = \frac{1}{2} \left\{ \left(\frac{\kappa^2 c^2 + 1}{c^2 \kappa} \right) N_1 - \frac{\kappa_1}{\kappa \lambda \cos\left(\frac{u}{c}\right)} N_2 \right\}. \tag{35}$$

where

$$\kappa = \sqrt{(\varphi')^2 + \varphi^2 \left((\alpha')^2 + \frac{1}{c^2} \right) + \frac{\lambda^2}{c^4} \left(1 - \frac{c^2}{\lambda^2} \right)}, \quad \kappa_1 = \varphi^2 \alpha', \tag{36}$$

and

$$\varphi = \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)}. \tag{37}$$

Corollary 8 *Let M be a generalized spherical surface of first kind given with the surface patch (30). If the second mean curvature H_2 vanishes identically then the angle function $\alpha(u)$ is a real constant.*

For any local surface $M \subset \mathbb{E}^4$ given with the regular surface patch $X(u, v)$ the normal curvature K_N is given with the following result.

Proposition 9 [4] *Let $M \subset \mathbb{E}^4$ be a local surface given with a regular patch $X(u, v)$ then the normal curvature K_N of the surface becomes*

$$K_N = \frac{g_{11}(L_{12}^1 L_{22}^2 - L_{12}^2 L_{22}^1) - g_{12}(L_{11}^1 L_{22}^2 - L_{11}^2 L_{22}^1) + g_{22}(L_{11}^1 L_{12}^2 - L_{11}^2 L_{12}^1)}{W^3}. \tag{38}$$

As a consequence of (33) with (38) we get the following result.

Corollary 10 *Any generalized spherical surface of first kind has flat normal connection, i.e., $K_N = 0$.*

Example 11 *In 1966, T. Otsuki considered the following special cases*

$$\begin{aligned} a) \quad f_1(u) &= \frac{4}{3} \cos^3\left(\frac{u}{2}\right), \quad f_2(u) = \frac{4}{3} \sin^3\left(\frac{u}{2}\right), \quad f_3(u) = \sin u, \\ b) \quad f_1(u) &= \frac{1}{2} \sin^2 u \cos(2u), \quad f_2(u) = \frac{1}{2} \sin^2 u \sin(2u), \quad f_3(u) = \sin u. \end{aligned}$$

For the case a) the surface is called Otsuki (non-round) sphere in \mathbb{E}^4 which does not lie in a 3-dimensional subspace of \mathbb{E}^4 . It has been shown that these surfaces have constant Gaussian curvature [16].

CASE III. For $n = 1$ and $m = 3$, the radius vector (25) satisfying the indicated properties describes the *generalized spherical surface* given with the radius vector

$$X(u, v) = \phi(u)\vec{e}_1 + \lambda \cos\left(\frac{u}{c}\right) \rho(v), \quad (39)$$

where

$$\phi(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} du, \quad (40)$$

and $\rho = \rho(v)$ parametrized by

$$\rho(v) = (g_1(v), g_2(v), g_3(v)),$$

$$\|\rho(v)\| = 1, \quad \|\rho'(v)\| = 1,$$

which lies on the unit sphere $S^2 \subset \mathbb{E}^4$. The spherical curve ρ has the following Frenet Frames;

$$\begin{aligned} \rho'(v) &= T(v) \\ T'(v) &= \kappa_\rho(v)N(v) - \rho(v) \\ N'(v) &= -\kappa_\rho(v)T(v). \end{aligned}$$

We call this surface a *generalized spherical surface of second kind*. Actually, these surfaces are the special type of meridian surface defined in [13], see also [5].

Proposition 12 *Let M be a meridian surface in \mathbb{E}^4 given with the parametrization (39). Then M has the Gaussian curvature*

$$K = -\frac{\kappa_\gamma \phi'(u)}{\lambda \cos\left(\frac{u}{c}\right)}, \quad (41)$$

where

$$\kappa_\gamma(u) = -\frac{\lambda}{c^2} \phi'(u) \cos\left(\frac{u}{c}\right) + \frac{\lambda}{c} \phi''(u) \sin\left(\frac{u}{c}\right).$$

Proof 13 Let M be a meridian surface in \mathbb{E}^4 defined by (39). Differentiating (39) with respect to u and v and we obtain

$$\begin{aligned} X_u &= \phi'(u) \vec{e}_1 - \frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \rho(v), \\ X_v &= \lambda \cos\left(\frac{u}{c}\right) \rho'(v), \\ X_{uu} &= \phi''(u) \vec{e}_1 - \frac{\lambda}{c^2} \cos\left(\frac{u}{c}\right) \rho(v), \\ X_{uv} &= -\frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \rho'(v), \\ X_{vv} &= \lambda \cos\left(\frac{u}{c}\right) \rho''(v). \end{aligned} \tag{42}$$

The normal space of M is spanned by

$$\begin{aligned} N_1 &= N(v), \\ N_2 &= -\frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \vec{e}_1 - \phi'(u) \rho(v), \end{aligned} \tag{43}$$

where $N(v)$ is the normal vector of the spherical curve ρ .

Hence, the coefficients of first and second fundamental forms are becomes

$$\begin{aligned} g_{11} &= \langle X_u(u, u), X_u(u, u) \rangle = 1, \\ g_{12} &= \langle X_u(u, v), X_v(u, v) \rangle = 0, \\ g_{22} &= \langle X_v(v, v), X_v(v, v) \rangle = \lambda^2 \cos^2\left(\frac{u}{c}\right), \end{aligned} \tag{44}$$

and

$$\begin{aligned} L_{11}^1 &= L_{12}^1 = L_{12}^2 = 0, \\ L_{22}^1 &= \kappa_\rho(v) \lambda \cos\left(\frac{u}{c}\right), \\ L_{11}^2 &= -\kappa_\gamma(u), \\ L_{11}^2 &= \phi'(u) \lambda \cos\left(\frac{u}{c}\right). \end{aligned} \tag{45}$$

respectively, where

$$\begin{aligned} \kappa_\gamma(u) &= f_1'(u) f_2''(u) - f_1''(u) f_2'(u) \\ &= -\frac{\lambda}{c^2} \phi'(u) \cos\left(\frac{u}{c}\right) + \frac{\lambda}{c} \phi''(u) \sin\left(\frac{u}{c}\right). \end{aligned}$$

Consequently, substituting (44)-(45) into (6) we obtain the result.

As a consequence of (45) with (38) we get the following result.

Proposition 14 *Any generalized spherical surface of second kind has flat normal connection, i.e., $K_N = 0$.*

Corollary 15 *Every generalized spherical surface of second kind is a meridian surface given with the parametrization*

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} du, \\ f_2(u) &= \lambda \cos\left(\frac{u}{c}\right). \end{aligned} \quad (46)$$

By the use of (40)-(41) with (46) we get the following result.

Corollary 16 *The generalized spherical surface of second kind has constant Gaussian curvature $K = 1/c^2$.*

As consequence of (7) we obtain the following result.

Proposition 17 *Let M be a generalized spherical surface of second kind given with the parametrization (39). Then the mean curvature vector of M becomes*

$$\vec{H} = \frac{1}{2f_2(u)} \{ \kappa_\rho(v) N_1 + (-\kappa_\gamma f_2(u) + f_1'(u)) N_2 \}. \quad (47)$$

where

$$\kappa_\rho(v) = \sqrt{g_1''(v)^2 + g_2''(v)^2 + g_3''(v)^2}.$$

Corollary 18 *Let M be a generalized spherical surface of second kind given with the parametrization (39). If*

$$\kappa_\gamma(u) = \frac{f_1'(u)}{f_2(u)}, \quad (48)$$

then M has vanishing second mean curvature, i.e., $H_2 = 0$.

Example 19 *Consider the curve $\rho(v) = (\cos v, \cos v \sin v, \sin^2 v)$ in $S^2 \subset \mathbb{E}^3$. The corresponding generalized spherical surface*

$$\begin{aligned} x_1(u, v) &= \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} du \\ x_2(u, v) &= \lambda \cos\left(\frac{u}{c}\right) \cos v \\ x_3(u, v) &= \lambda \cos\left(\frac{u}{c}\right) \cos v \sin v \\ x_4(u, v) &= \lambda \cos\left(\frac{u}{c}\right) \sin^2 v. \end{aligned} \quad (49)$$

is of second kind.

References

- [1] K. Arslan, B. Bayram, B. Bulca and G. Öztürk, *General rotation surfaces in E^4* , Results. Math., (2012), DOI 10.1007/s00025-011-0103-3.
- [2] B. Bulca, K. Arslan, B.K. Bayram and G. Öztürk, *Spherical product surfaces in \mathbb{E}^4* . An. St. Univ. Ovidius Constanta, 20(2012), 41-54.
- [3] B. Bulca, K. Arslan, B.K. Bayram, G. Öztürk and H. Ugail, *Spherical product surfaces in \mathbb{E}^3* . IEEE Computer Society, Int. Conference on CYBERWORLDS, 2009.
- [4] B. Bulca, *\mathbb{E}^4 deki Yüzeylerin Bir Karakterizasyonu*, PhD.Thesis, Bursa, 2012.
- [5] K. Arslan, B. Bulca, and V. Milousheva, *Meridian Surfaces in \mathbb{E}^4 with Pointwise 1-type Gauss map*. Bull. Korean Math. Soc., 51(2014), 911-922.
- [6] B.Y., Chen, *Geometry of Submanifolds*, Dekker, New York, 1973.
- [7] B. Y. Chen, *Pseudo-umbilical surfaces with constant Gauss curvature*, Proceedings of the Edinburgh Mathematical Society (Series 2), 18(2) (1972), 143-148.
- [8] B.-Y. Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, 1981.
- [9] D.V. Cuong, *Surfaces of Revolution with Constant Gaussian Curvature in Four-Space*, arXiv:1205.2143v3.
- [10] DeSmet, P.J., Dillen, F., Verstrelen, L., Vrancken, L., *A pointwise inequality in submanifold theory*. Arch. Math.(Brno) 35(1999), 115-128.
- [11] Dursun, U. and Turgay, N.C. *General rotational surfaces in Euclidean space \mathbb{E}^4 with pointwise 1-type Gauss map*. Math. Commun., 17(2012), 71-81.
- [12] G. Ganchev and V. Milousheva, *On the Theory of Surfaces in the Four-dimensional Euclidean Space*. Kodai Math. J. **31** (2008), 183-198.
- [13] G. Ganchev and V. Milousheva, *Invariants and Bonnet-type theorem for surfaces in \mathbb{R}^4* , Cent. Eur. J. Math., 8 (2010), no. 6, 993-1008.
- [14] V. A. Gor'kavyi and E. N. Nevmerzhitskaya, *Two-dimensional Pseudo-spherical surfaces with degenerate bianchi transformation*, Ukrainian Mathematical Journal, Vol. 63(2012), No. 11, Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 63(2011), No. 11, pp. 1460-1468.
- [15] N.H. Kuiper, *Minimal Total Absolute Curvature for Immersions*. Invent. Math., 10(1970), 209-238.
- [16] T. Otsuki, *Surfaces in the 4-dimensional Euclidean Space Isometric to a Sphere*, Kodai Math. Sem. Rep. **18**(1966), 101-115.

- [17] V. Velickovic, *On Surface of Rotation of a given Constant Gaussian Curvature and Their Visualization*, Preprint.
- [18] Y.C. Wong, *Contributions to the theory of surfaces in 4-space of constant curvature*, Trans. Amer. Math. Soc, **59** (1946), 467-507.
- [19] D.W. Yoon, *Some Properties of the Clifford Torus as Rotation Surfaces*, Indian J. Pure Appl. Math. 34(2003), 907-915.

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